THE DECEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-second regular meeting of the San Francisco Section, postponed from October 26, was held at the University of California on Saturday, December 14, 1918. The total attendance was thirteen, including the following nine members of the Society:

Professor B. A. Bernstein, Professor Florian Cajori, Professor M. W. Haskell, Professor Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Dr. F. R. Morris, Professor C. A. Noble, and Dr. Pauline Sperry.

The chairman, Professor Manning, opened the meeting, later giving way to Professor Cajori, the newly elected chairman. At the election of officers the following were chosen for one year: Chairman, Professor Florian Cajori; Secretary, Professor B. A. Bernstein; Programme committee, Professor L. M. Hoskins, Professor W. A. Manning, and Professor B. A. Bernstein.

It was decided to hold the next fall meeting on Saturday, October 25, 1919, at the University of California.

The following papers were read:

(1) Professor FLORIAN CAJORI: "A note on the history of Playfair's axiom."

(2) Professor FLORIAN CAJORI: "On the Aristotelian tract, De lineis insecabilibus."

(3) Professor M. W. HASKELL: "Triangles inscribed and circumscribed to a plane cubic."

(4) Professor D. N. LEHMER: "A non-tentative method of solving the indeterminate equation $Ax + By + Cz + \cdots = n$."

(5) Professor E. T. BELL: "A partial isomorph of trigonometry."

(6) Professor E. T. BELL: "Arithmetical paraphrases (I)."

(7) Professor E. T. BELL: "Arithmetical paraphrases (II); elliptic and theta series."

(8) Professor E. T. BELL: "Arithmetical paraphrases (III); chiefly for the doubly periodic functions of the first, second, and third kinds."

(9) Professor E. T. BELL: Arithmetical paraphrases (IV); class number formulas."

Professor Bell's papers were read by title. Abstracts of the papers follow below.

1. Professor Cajori points out that Playfair's axiom on parallel lines, now usually attributed to William Ludlam (1785), was used by Joseph Fenn in Dublin, in 1769.

2. The De lineis insecabilibus enumerates 5 arguments, current in Aristotle's day, in favor of the existence of indivisible lines, 22 considerations showing the invalidity of those arguments, and 26 considerations tending to disprove the view that a line is made up of points. Some of the arguments are rigorous. In Professor Cajori's opinion the tract deserves a place in the history of mathematics.

3. Professor Haskell shows that there are twenty-four triangles inscribed and circumscribed to a non-singular cubic. These twenty-four triangles occur in four sets of six each, corresponding to the four inflexional triangles, so that in each set three are triply in perspective with the other three, the centers of perspective for each pair being three collinear inflexions. The vertices of each pair lie on a conic.

For a cubic with a double point, the total number of triangles is two, which are real if the double point is isolated and imaginary otherwise. A cuspidal cubic has no such triangle.

Further, with respect to each inflexional line there is a six-point involution on the cubic, determined by a pencil of conics, where each set of six points are the vertices of two triangles triply in perspective with respect to the inflexions lying on the given line. These two triangles become coincident in three cases, the vertices then being sextactic points.

4. The solution of the indeterminate equation $Ax + By = n$ by means of the regular continued fraction representing A/B furnishes an example of the very few non-tentative methods available in the theory of numbers. Professor Lehmer uses the theory of continued fractions of higher orders to obtain all the solutions of the equation $Ax + By + Cz + \cdots = n$.

The paper will be offered to the *American Journal* as part of a memoir on continued fractions of higher orders, an abstract of which has already been inserted in the *Proceedings of the National Academy of Sciences*.

5. Professor Bell defines a function to be even or odd in a set of variables according as it does not or does change sign when the signs of all the variables are changed simultaneously. Let X_i, Y_j denote sets containing a_i, b_j variables respectively; then

$$f(X_1, X_2, \cdots, X_r|Y_1, \cdots, Y_s)$$

is defined to be even in each X, odd in each Y, and its parity is the symbol

$$p(a_1, a_2, \cdots, a_r|b_1, b_2, \cdots, b_s)$$

whose significance is obvious. If

$$a_i = 1 = b_j, (i = 1, \cdots, r; j = 1, \cdots, s),$$

the parity is written

$$p(1^r|1^s).$$

It is shown that any function whose parity is

$$p(a_1, a_2, \cdots, a_r|b_1, b_2, \cdots, b_s)$$

is linearly expressible in terms of 2^κ properly chosen functions F, whose parities are all of the form

$$p(1^\alpha|1^\beta),$$

where

$$\alpha + \beta = \kappa,$$

and

$$\kappa = (a_1 + a_2 + \cdots + a_r) + (b_1 + b_2 + \cdots + b_s) - (r + s).$$

An arbitrary function (one neither odd nor even in any set of its variables) of k variables is similarly expressible, the number of the F being 2^k. These results are useful in the arithmetical applications of the elliptic and theta functions.

6. From 1858 to 1865 Liouville published intermittently a series of eighteen memoirs, "Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres," wherein he gave without proof many remarkable theorems which he and others applied to various questions of interest, including class number formulas. Proofs by extensions of the method used by Dirichlet in proving Jacobi's four square theorem have been given for most of Liouville's general results by H. J. S. Smith (1865), T. Pepin (1889–90), G. B. Mathews

(1892), and E. Meissner (1907). But, as remarked by Bachmann (1910): "Eine zusammenhängende systematische Herleitung und Verbindung dieser Formeln (Liouville's), welche ihren Quell und die Prinzipien klarlegte, nach denen die in den Formeln auftretenden Argumente der Funktionen, auf welche sie sich beziehen, zu wählen sind, würde sehr wertvoll sein." By means of the theorems in his first paper and another simple principle, Professor Bell shows that the classical expansions in the theory of elliptic and theta functions may be paraphrased directly into theorems of the Liouville kind; moreover that a simple method is thereby provided for the discovery at will of new results of the same general sort. Liouville's formulas are nearly all paraphrases of Jacobi's expansions in the Fundamenta Nova and in the memoirs on rotation.

7. In his second paper on paraphrases Professor Bell prepares and collects elliptic and theta expansions in a form suitable for paraphrase, and gives sixteen trigonometric identities basic for Liouville's numerous (unproved) theorems on the quadratic forms of certain primes. The series are classified according to the forms of the divisors d, δ appearing in the sine, cosine coefficients of the powers of q. Similar series for sets of non-doubly periodic theta quotients, such as Biehler's, Hermite's, and Humbert's, are also given.

8. In his third paper on paraphrases, Professor Bell applies the methods of his first to a few of the series in the second. Among other results it is shown that all the formulas in Liouville's fifth memoir (and many more) are immediate paraphrases of a few classic formulas relating to the doubly periodic functions of the second kind. Illustrative of the simplicity of the method, the formulas of Liouville's sixth memoir follow at once from Jacobi's series for $\mathrm{sn}^3 u$.

9. Professor Bell's fourth paper on paraphrases applies the method to the series for non-doubly periodic theta quotients given by Hermite (1862) and extended by G. Humbert (1907). The resulting paraphrases involve class numbers and arbitrary functions odd or even in one or more variables. It is shown how such paraphrases are related to the number of representations of an integer as a sum of 3, 5, 7, 9, 11, $\cdots$ squares.

B. A. BERNSTEIN,
Secretary of the Section.

THE SCIENTIFIC WORK OF MAXIME BÔCHER.

BY PROFESSOR GEORGE D. BIRKHOFF.

WITH the recent death of Professor Maxime Bôcher at only fifty-one years of age American mathematics has suffered a heavy loss. Our task in the following pages is to review and appreciate his notable mathematical work.*

His researches cluster about Laplace's equation $\Delta u = 0$, which is the very heart of modern analysis. Here one stands in natural contact with mathematical physics, the theory of linear differential equations both total and partial, the theory of functions of a complex variable, and thus directly or indirectly with a great part of mathematics.

His interest in the field of potential theory began in undergraduate days at Harvard University through courses given by Professors Byerly and B. O. Peirce. There is still on file at the Harvard library an undergraduate honor thesis entitled "A thesis on three systems of parabolic coördinates," written by him in 1888. Under the circumstances it was inevitable that he should use formal methods in dealing with his topic, but a purpose to penetrate further is found in the concluding sentences. No better opportunity for fulfilling such a purpose could have been granted than was given by his graduate work under Felix Klein at Göttingen (1888–1891).

In the lectures on Lamé's functions which Klein delivered in the winter of 1889–1890 his point of departure was the cyclidic coördinate system of Darboux. This sytem of coordinates was known to be so general as to include nearly all of the many types of coördinates useful in potential theory, and Wangerin had shown (1875–1876) how solutions of Laplace's equation existed in the form of triple products, each factor being a function of one of the three cyclidic coördinates. After presenting this earlier work Klein extended his "oscillation theorem" for the case of elliptic coördinates (1881) to the more general cyclidic coördinates. By this means he was able to attack the problem of setting up a potential function taking on given values over the surface of a solid bounded by

* An account of his life and service by Professor Osgood will appear in a later number of the BULLETIN.

six or fewer confocal cyclides. This function was given by a series of the triple "Lamé's" products discovered by Wangerin.

Klein also aimed to get at the various forms of series and integrals previously employed in potential theory as actual limiting cases, and thus to bring out the underlying unity in an extensive field of mathematics.

The task which Bôcher undertook was to carry through the program sketched by Klein. He did this admirably in his first mathematical paper "Ueber die Reihenentwickelungen der Potentialtheorie," which appeared in 1891 and which served both as a prize essay and as his doctor's dissertation at Göttingen.* But the space available was so brief that he was only able to outline results without giving their proofs.

One must look to his book with the same title,† published three years later, for an adequate treatment of the subject. Here is also to be found original work not outlined in his dissertation. It was characteristic that he did not call attention explicitly to the new advances although these formed his most important scientific work in the years 1891–1894. We turn now to a consideration of this book, which thus contains nearly all that he did before 1895.

Besides giving the classification of all types of confocal cyclides in the real domain and of the corresponding Lamé's products, as sketched by Klein, Bôcher determined to what extent the theorem of oscillation holds in the degenerate cases and found an interesting variety of possibilities.

The difficulties presented by these degenerate cases are decidedly greater than those of the general case when the singular points e_i $(i = 1, 2, 3, 4, 5)$ of the Lamé's linear differential equation are regular with exponents 0, 1/2. A very simple degenerate case is that arising when two such points coincide in a single point and one of the two intervals (m_1, m_2), (n_1, n_2) under consideration ends at this point. By an extension of Klein's geometric method, he proved that the theorem of oscillation fails to hold even here.

More specifically, the facts are as follows. In the general case the oscillation theorem states that for any choice of integers m, n $(m, n \geqq 0)$ there is a unique choice of the two ac-

* This paper appears as (2) in the chronological list of papers given at the end of the present article. Hereafter footnote references to papers will be made by number.

† (15).

cessory parameters in the differential equation, yielding solutions u_1, u_2 such that u_1 vanishes at m_1 and m_2, and m times for $m_1 < x < m_2$, while u_2 vanishes at n_1 and n_2, and n times for $n_1 < x < n_2$. If now, for instance, m_1 lies at the double singular point $e_1 = e_2$, while $m_1 < m_2 < e_3 < e_4 < n_1 < n_2 < e_5$, there exist such solutions u_1, u_2 only if $n > r_m$ where r_m is an integer increasing indefinitely with m. But, to compensate for this deficiency of solutions of the boundary value problem, Bôcher found it necessary to introduce solutions u_{1k}, u_{2k} dependent on n and a continuous real parameter k such that u_1 vanishes at m_2 and infinitely often for $m_1 < x < m_2$ although remaining finite, while u_2 vanishes at n_1 and n_2, and n times for $n_1 < x < n_2$.

The corresponding expansion in Lamé's products presents a remarkable form under these circumstances, for it is made up of a series and an integral component. In another case this type of expansion takes the form of an integral augmented by a finite number of complementary terms, as he had pointed out in an important paper "On some applications of Bessel's functions with pure imaginary index,"* published in 1892 in the *Annals of Mathematics*.

Although dealing satisfactorily with the oscillation theorem in the case specified above and other similar cases, Bôcher did not discuss adequately the case in which three or more singular points unite to form an irregular singular point.† Indeed it appears that he fell into an error of reasoning as follows. If the irregular point be taken at $t = +\infty$ the Lamé's equation has the form

$$\frac{d^2y}{dt^2} = \varphi y,$$

where in the case under consideration φ has a limit $\varphi_0 \neq 0$ as t becomes infinite. The lemma which Bôcher then sought to prove‡ was that there always exists a solution y *finite* for $t \geqq T$ and not identically zero. His proof for the case $\varphi_0 > 0$ is essentially correct. Here he interpreted the equation above as the equation of motion of a particle distant y from a point O of its line of motion and repelled from it with a force φy.

* (7). In passing, attention may also be called to a slightly earlier article (3) on Bessel's functions.

† See (15), p. 179.

‡ See (15), p. 177.

The gist of the argument employed is that one can find an initial velocity of projection toward O just sufficient to carry it into that point as a limiting position. This part of the lemma constitutes a very simple and interesting theorem concerning a special type of irregular point. In the case $\varphi_0 < 0$, however, using a similar dynamical interpretation, he argued* "we have infinitely many oscillations as we approach $t = +\infty$, and since the attractive force is not infinitely weak, the amplitudes of the oscillations remain finite." This argument appears insufficient although the lemma as stated for Lamé's equation is probably correct.† To satisfactorily complete the discussion it would seem to be necessary to call in the explicit analytic theory of the irregular singular point, since the corresponding theory of the regular singular point is required in the simpler cases.‡

In his book Bôcher considered the boundary problem under *periodic* conditions, when the interval between two adjacent singular points is taken an even number of times and is regarded as closed; this case arises, for example, when the solid in the potential problem is a complete ellipsoid. Here the function φ in the linear differential equation above written is an even doubly periodic function with real period. By the aid of these properties of φ he reduced the new boundary problem to one of the ordinary type.

Likewise in treating the roots of Lamé's polynomials he made a distinct advance by extending the dynamical method of Stieltjes from the real axis to the complex plane. Thus he was able to prove that the roots of these polynomials lie within the triangle whose vertices are the three finite singular points of the corresponding Lamé's equation.

Finally we may note that at the end of his book he obtained all Lamé's products satisfying the equation $\Delta u + k^2 u = 0$.

The determinative effect of the dissertation and book upon the direction of Bôcher's later researches was very great. In the first place he had used sphere geometry and the algebra of elementary divisors as essential tools in analysis; his resulting interest in the fundamental parts of geometry and algebra never subsided, and some of his research lies in these fields.

* (15), p. 178. The translation is not literal.

† In this connection see (7), p. 150, footnotes.

‡ Since the above was written Professor Osgood has disposed of the question at issue by elementary means. See his note in this number of the BULLETIN.

But, more important still, he was brought into contact with open mathematical questions. The most vital of these questions from the purely mathematical point of view was doubtless the very difficult analytical question of convergence and representation presented by the series of Lamé's products. This was the outstanding problem which Klein emphasized,* but to which Bôcher seems never to have given particular attention. Another more practical direction of effort was afforded by the task of giving rigorous and accessible form to the work of Sturm and Klein on the real solutions of ordinary linear differential equations and then going on further in this overlooked but attractive field of research. It was primarily to this task that he now turned.

In 1897 he published an article in the BULLETIN† showing the immediate usefulness of Sturm's theorems for fixing the distribution of the roots of Bessel's functions with real index. A year later in the same place he presented the fundamentals of Sturm's work in simplified rigorous form, and gave the first analytic proof of Klein's theorem of oscillation.‡

Reference should also be made to his article on the boundary problems of ordinary differential equations which appeared in the German mathematical encyclopedia in 1900. This article together with his address on "Boundary problems in one dimension" before the Fifth International Congress of Mathematicians in 1912 give an excellent account of this field to the latter date.

Bôcher wrote a considerable number of other papers in this same field.§ Perhaps the most important of these are the three to which we will refer first and which appeared in the beginning volumes of the *Transactions*.

His paper "Application of a method of d'Alembert to the proof of Sturm's theorems of comparison" (1900) contained an elegant proof of what Bôcher had called the theorems of comparison. His method was entirely different from Sturm's, being based on the Riccati's resolvent equation, and was very simple.

In the second of these papers "On certain pairs of trans-

* See the concluding pages of his 1889–1890 lectures on Lamé's functions and his preface to Bôcher's book (15).

† (26). See (38) also.

‡ (30), (31), (35).

§ (32), (38), (42), (46), (48), (49), (50), (55), (65), (81), (85), (92), (93), (100).

cendental functions whose roots separate each other"* (1901) his starting point was the linear differential equation

$$y'' + py' + qy = 0,$$

and a pair of linear forms in y, y',

$$\Phi = \varphi_2 y' - \varphi_1 y, \quad \Psi = \psi_2 y' - \psi_1 y.$$

These latter satisfy a "homogeneous Riccati's equation"

$$(\varphi_1\psi_2 - \varphi_2\psi_1)(\Phi'\Psi - \Phi\Psi') + A\Phi^2 + B\Phi\Psi + C\Psi^2 = 0,$$

and Bôcher considered the relation of the roots of Φ, Ψ.

He notes first that Φ, Ψ cannot vanish together unless $\varphi_1\psi_2 - \varphi_2\psi_1 = 0$, for otherwise $y = y' = 0$. In order that Φ, Ψ cannot vanish together it is thus sufficient to assume $\varphi_1\psi_2 - \varphi_2\psi_1 \neq 0$. Also if $\Phi = 0$, then $\Phi' \neq 0$ if $C \neq 0$, by the above equation. A like remark holds for Ψ. Hence the roots of Φ, Ψ are simple if $A \neq 0$, $C \neq 0$.

Under these hypotheses between any pair of adjacent roots of Φ there must be a root of Ψ. For if Ψ has no such root the homogeneous Riccati's equation at these roots shows that Φ' has one and the same sign at both roots, which is impossible. Likewise between any pair of adjacent roots of Ψ there must be a root of Φ.

Hence *the roots of* Φ, Ψ *separate each other if*

$$\varphi_1\psi_2 - \varphi_2\psi_1 \neq 0, \; A \neq 0, \; C \neq 0.$$

This is the third theorem of the paper. The sixth theorem gives similar conditions sufficient to ensure cyclical separation of the roots of three linear forms.

Here Bôcher not only achieved greater generality and simplicity than Sturm but, as I wish to point out, he has reached a maximum of generality.

For, let y_1, y_2 be any pair of linearly independent solutions yielding the values Φ_1, Φ_2 and Ψ_1, Ψ_2 of Φ and Ψ. Then

$$\Phi = c_1\Phi_1 + c_2\Phi_2, \quad \Psi = c_1\Psi_1 + c_2\Psi_2$$

are the general values of Φ, Ψ. If Φ_1, Φ_2 are regarded as the homogeneous coordinates of a point P in the projective line, Φ vanishes if P coincides with $E \equiv (-c_2, c_1)$; similarly Ψ vanishes if $Q = (\Psi_1, \Psi_2)$ coincides with the same point E.

* See also (100).

Clearly the roots of Φ, Ψ will only be distinct for all values of c_1, c_2 if $\varphi_1\psi_2 - \varphi_2\psi_1 \neq 0$. Moreover, if these roots are to separate each other for all values of c_1, c_2, the points P, Q must pass *any point* E in alternation. This is only possible if P, Q never reverse their direction of motion; in other words the Wronskians of Φ_1, Φ_2 and of Ψ_1, Ψ_2 must be of invariant signs. Taking into account the fact that $y_1y_2' - y_1'y_2$ is not zero, this gives precisely the conditions $A \neq 0$, $C \neq 0$.

This same geometric interpretation shows a similar generality in the other theorems.

Of like completeness is the third paper "On the real solutions of systems of two homogeneous linear differential equations of the first order" (1902), where he treated analogous questions and also derived comparison theorems.

It was a matter of primary interest with him to vary proofs of known theorems as well as to discover new theorems. An illustration in point is afforded by his treatment of the elementary separation theorem for the roots of linearly independent solutions y_1, y_2 of an ordinary linear differential equation of the second order.

Here he first gave a very brief proof* based on the function y_1/y_2: if y_1 vanishes at a and b but not for $a < x < b$, while y_2 is not zero for $a \leqq x \leqq b$, then the derivative of y_1/y_2 is of one sign for $a < x < b$ since $y_1y_2' - y_1'y_2 \neq 0$. This is impossible. By this argument and a like argument based on y_2/y_1 it follows that the roots of y_1, y_2 separate each other. In the same place† he isolates a geometric proof implicitly given by Klein depending on the fact that if y_1, y_2 be taken as homogeneous coördinates of a point in the projective line then $y_1y_2' - y_1'y_2 \neq 0$ is the condition that this point moves continually in one sense. Later he gave a second analytic proof based on the function

$$\frac{y_1'}{y_1} - \frac{y_2'}{y_2},$$‡

and also a second geometric proof* based on the vector $y_1 + \sqrt{-1}\,y_2$ in the complex plane which will rotate continually in one sense if $y_1y_2' - y_2y_1' \neq 0$.

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* (26), p. 210.
† Footnote, p. 210.
‡ (48).
§ (99), pp. 46–47.

It was not easy for him to believe that the methods of Sturm were inadequate to deal with any particular boundary problem in one dimension. The problem for periodic conditions, which had been formulated by him in his encyclopedia article, was first successfully attacked by Mason in 1903–1904 by means of the calculus of variations. In a very interesting note published in 1905,* Bôcher showed that the principal result fell out immediately by the methods of Sturm, and that these methods were applicable under much more general conditions. Likewise in his address before the Fifth International Congress of Mathematicians alluded to above he noted that the equation

$$\frac{d}{dx}\left(k\frac{du}{dx}\right) + (\lambda g - l)u = 0, \qquad l < 0,$$

(λ a parameter) comes directly under the case treated by Sturm after division by $|\lambda|$ even if g changes sign. This simple remark disposed of the necessity of treating this case separately, as had been done earlier.

Bôcher was interested in all phases of the theory of ordinary linear differential equations with real independent variable. Having seen the gap in the theory of the regular singular point for real independent variable when the coefficients are not analytic, he proved that theorems analogous to those given by Fuchs in the complex domain are true.† It was necessary here to replace the power series treatment by a variation of the method of successive approximation which has been seen later to afford a new approach to the theory of the regular singular point in the complex domain.

He also did some work in the field of fundamental existence theorems for linear differential equations.‡ He showed that it is sufficient to impose the condition of integrability (joined with other conditions) upon the coefficients in place of Peano's condition of continuity,§ and thus advanced beyond Peano. Bôcher seems also to have been the first to prove that the solutions of a linear differential system are continuous functionals of the coefficients.‖

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* (65).
† (37), (40), (41).
‡ (32), (37), (56).
§ (56), p. 311.
‖ (56), p. 315; (55), p. 208.

In 1901 he published a paper on "Green's functions in space of one dimension," in which he pointed out that the Green's function for the equation of Laplace in one dimension $y'' = 0$, exhibited by Burkhardt in 1894, might be extended to the general nth order ordinary linear differential equation with fairly general boundary conditions. These extended Green's functions have turned out to be of great importance. Later he returned to the subject of Green's functions with the most general linear boundary conditions and set up these functions for linear difference equations.* Also he extended the notion of adjoint boundary conditions to very general cases.†

We have now referred briefly to the most important of his researches on ordinary linear differential equations with *real* independent variable. In this domain his best work is perhaps to be found. Directly springing from this field were his researches on linear dependence of functions of a single real variable‡—an important topic which he was the first to isolate sufficiently from the field of linear differential equations.

His paper on "The roots of polynomials which satisfy certain linear differential equations of the second order"§ lies in the field of ordinary linear differential equations with a *complex* variable. Here he generalizes further the extension of the method of Stieltjes which he had employed in dealing with Lamé's polynomials.

The series arising in mathematical physics had been Bôcher's point of departure. Indeed it is the existence of these series which constitutes the main importance of the boundary value problems of linear differential equations. Nevertheless he gave special attention only to Fourier's series which he took up in an expository article in the *Annals of Mathematics* for 1906.‖ Here he called attention to the remarkable phenomenon exhibited by a Fourier's series near a point of discontinuity, previously noted by Gibbs and called "Gibbs's phenomenon" by Bôcher who gave the first adequate treatment of it.¶

His contributions to the theory of the harmonic function in two dimensions are elegant and distinctly important.

* (81).
† (85).
‡ (43), (45), (47), (51), (97).
§ (29).
‖ (67). See also (89).
¶ Reference may also be made here to the short note on infinite series (60).

The first of these occurs incidentally in his paper "Gauss's third proof of the fundamental theorem of algebra."* It consists in a proof of the average value theorem by means of Gauss's theorem for the circle, which in polar coordinates r, φ is

$$\int_0^{2\pi} \frac{\partial u}{\partial r} d\varphi = 0.$$

Integrating with respect to r from 0 to a and reversing the order of integration, we get

$$\int_0^{2\pi} (u(a, \varphi) - u(0, \varphi)) d\varphi = 0,$$

whence the average value theorem follows at once. This very neat proof was probably suggested by the artifice used by Gauss in his third proof of the fundamental theorem of algebra.

The "Note on Poisson's integral" (1898) gives a more natural interpretation of Poisson's integral than had been stated before. By the average value theorem a harmonic function is the average of its values on any circle with its center at the given point. He generalized this theorem in the spirit of the geometry of inversion and thus reached a visual interpretation of Poisson's integral which may be formulated as follows: The value of a harmonic function at any point within a circle is the average of its values as read by an observer at the point who turns with uniform angular velocity, if the rays of light to his eye take the form of circular arcs orthogonal to the given circle.

According to Riemann's program, the theory of harmonic functions requires a development independent of the theory of functions of a complex variable. In 1905 Bôcher demonstrated† that a harmonic function could not become infinite at a point unless it was of the form $C \log r + v$, where C is a constant, r is the distance from a variable point to the given point and v is harmonic at that point. This theorem corresponds to the fundamental theorem in functions of a complex variable which states that if $f(z)$ becomes infinite at the isolated singular point $z = a$, then $f(z)$ is of the form $(z - a)^{-r} g(z)$ where r is a positive integer and $g(z)$ is analytic and not zero

* (17), p. 206.
† (59).

at $z = a$. He demonstrated further that a similar theorem holds for large classes of linear partial differential equations.

Another extremely interesting paper "On harmonic functions in two dimensions" appeared in 1906. Here he defines u to be harmonic if it is single valued and continuous with continuous first partial derivatives and satisfies Gauss's theorem for every circle. If u possessed continuous second partial derivatives also it would then follow at once by Green's theorem that u is harmonic in the customary sense. But it is the merit of Bôcher's paper to have proved that u is harmonic in the ordinary sense without further assumptions. On the basis of the definition made, the average value theorem is first deduced as outlined above. Also if s', n' are the new variables s, n after an inversion (taking circles into circles) we have

$$0 = \int \frac{\partial u}{\partial n} ds = \int \frac{\partial u}{\partial n'} ds'$$

along corresponding circles, since $ds'/dn' = ds/dn$ (the inversion being conformal). Thus u is "harmonic" in the transformed plane also, so that the definition is invariant under inversion. Hence Poisson's integral formula, which comes from the average value theorem by inversion, also holds, and u is harmonic in the ordinary sense.

He also determined the precise region of convergence of the real power series in x, y for any harmonic function $u(x, y)$.*

In connection with his papers on harmonic functions in two dimensions it is natural to call to mind his early paper "On the differential equation $\Delta u + k^2 u = 0$" (1893), which is taken in two dimensions. The "u-functions" so defined give a generalization of harmonic functions which he treated by means of the fact that $u(x, y)e^{kz}$ satisfies Laplace's equation in three dimensions. A similar method had been employed earlier by Klein.

Practically none of Bôcher's work lies directly in the field of functions of a complex variable.†

We have still to consider his contributions in the fields of algebra and geometry. In the early paper on the fundamental theorem of algebra cited above he made clear how, by taking for granted a few theorems in functions of a complex variable,

* (74).

† See (78), however.

an immediate proof could be given; and then he went on to show that by elimination of these theorems, the proof could be given a second more fundamental form and finally a third form due to Gauss and involving only distinctly elementary theorems. In a second paper* he simplified Gauss's proof very considerably by replacing Gauss's auxiliary function xf'/f by $1/f$. Here $f = 0$ is the given equation.

Here and elsewhere he succeeded in simplifying an apparently definitive proof. This kind of work was congenial to Bôcher, who believed that mathematics was capable of almost indefinite simplification, and that such simplification was of the highest consequence.

In the paper with the title "A problem in statics and its relation to certain algebraic invariants" (1904) he employed a dynamical method similar to his extension of the method of Stieltjes in order to develop an interpretation of the roots of covariants as the positions of equilibrium of particles in the complex plane. Thus if f_1, f_2 are polynomials of the same degree in the homogeneous variables x_1, x_2, the vanishing of their Jacobian determines the points of equilibrium in the field of force under the inverse first power law due to particles of "mass" 1 at the roots of f_1 and of "mass" -1 at the roots of f_2 in the x_1/x_2 plane.

We shall not refer to his geometrical papers† save to mention the one entitled "Einige Sätze über projective Spiegelung" (1893) in which he proves that conics in different planes may be projectively reflected into each other through a pair of lines in four ways, and also that the general collineation of space may be represented as the product of a rigid motion and a projective reflection through a pair of lines.

Besides this original research he undertook various more or less didactic articles with characteristic unselfishness.‡ However, just as in the article on Fourier's series, matter of an original cast is nearly always present.

The same may be said of his books,§ even of the most elementary. We have already considered his book on the series of potential theory. Of the others, the most significant are his Algebra, where a satisfactory exposition of the elementary

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* (18).
† (6), (8), (12), (13), (53).
‡ (14), (20), (24), (39), (66), (67), (70), (73), (83), (92).
§ (15), (71), (77), (94), (95), (99).

divisor theory is given, his Cambridge tract on integral equations,* and his Paris 1913–14 lectures "Leçons sur les Méthodes de Sturm." In the last is given the first complete discussion of the convergence of the series used in the method of successive approximations. This furnishes another good instance of Bôcher's power to seize on important theorems which have been missed although near at hand. In concluding this brief survey it is worth while noting that a few of his papers are fairly popular in character.†

In a recent one of these, "Mathématiques et mathématiciens Français" (1914), while speaking of the characteristics of American creative work in all fields (page 9), Bôcher says "Ce qu'il y a de plus caractéristique dans la meilleure production intellectuelle américaine, c'est la finesse et le contrôle voulu des moyens et des effets. La faute la plus commune dans ce que nous avons fait de mieux, ce n'est pas l'excès de force, mais plutôt son défaut" and later (page 10) "Ce que je viens de dire se rapporte aussi bien aux mathématiques qu'à toute autre branche de la production intellectuelle en Amérique." There can be no doubt that this characterization is applicable to his own mathematical production. His papers excel in simplicity and elegance, and nearly all of them treat subjects of great importance to marked advantage. The *usefulness* of his papers is exceptional,‡

In amount and quality his production exceeds that of any American mathematician of earlier date in the field of pure mathematics.

Because of this fact and the weight he has added to our mathematical traditions in other ways, Maxime Bôcher will ever remain a memorable personality in American mathematics.

List of Bôcher's Writings.‖

1888.

(1) The meteorological labors of Dove, Redfield and Espy. *American Meteorological Journal*, vol. 5, No. 1, pp. 1–13, May.

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* In connection with this, attention should be called to a short note on integral equations listed as (84) below.

† (1), (9), (11), (82), (90), (91). His first paper "On the meteorological labors of Dove, Redfield and Espy" was a youthful essay written about the time of his graduation from Harvard University.

‡ This is brought out clearly in Professor Osgood's Lehrbuch der Funktionentheorie, vol. 1.

‖ Substantially as compiled by him.

1891.

(2) Über die Reihenentwickelungen der Potentialtheorie. Gekrönte Preisschrift und Dissertation. Göttingen, Kästner. 4 + 66 pp.

1892.

(3) On Bessel's functions of the second kind. *Annals of Mathematics*, vol. 6, No. 4, pp. 85–90, Jan.
(4) Pockels on the differential equation $\Delta u + k^2 u = 0$ [Review]. *Annals of Mathematics*, vol. 6, No. 4, pp. 90–92, Jan.
(5) Geometry not mathematics [Letter to editor]. *Nation*, vol. 54, No. 1390, p. 131, Feb.
(6) On a nine-point conic. *Annals of Mathematics*, vol. 6, No. 5, p. 132, March.
(7) On some applications of Bessel's functions with pure imaginary index. *Annals of Mathematics*, vol. 6, No. 6, pp. 137–160, May.
(8) Note on the nine-point conic. *Annals of Mathematics*, vol. 6, No. 7, p. 178, June.
(9) Collineation as a mode of motion. *Bulletin of the New York Mathematical Society*, vol. 1, No. 10, pp. 225–231, July.

1893.

(10) On the differential equation $\Delta u + k^2 u = 0$. *American Journal of Mathematics*, vol. 15, No. 1, pp. 78–83, Jan.
(11) A bit of mathematical history. *Bulletin of the New York Mathematical Society*, vol. 2, No. 5, pp. 107–109, Feb.
(12) Some propositions concerning the geometric representation of imaginaries. *Annals of Mathematics*, vol. 7, No. 3, pp. 70–72, March.
(13) Einige Sätze über projective Spiegelung. *Mathematische Annalen*, vol. 43, No. 4, pp. 598–600.
(14) Chapter IX, Historical Summary, pp. 267–275. An Elementary Treatise on Fourier's Series and Spherical, Cylindrical and Ellipsoidal Harmonics. By W. E. Byerly. Boston, Ginn.

1894.

(15) Über die Reihenentwickelungen der Potentialtheorie. Mit einem Vorwort von Felix Klein. Leipzig, Teubner, 8 + 258 pp.

1895.

(16) Hayward's Vector Algebra [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 1, No. 5, pp. 111–115, Feb.
(17) Gauss's third proof of the fundamental theorem of algebra. *Bulletin of the American Mathematical Society*, ser. 2, vol. 1, No. 8, pp. 205–209, May.
(18) Simplification of Gauss's third proof that every algebraic equation has a root. *American Journal of Mathematics*, vol. 17, No. 3, pp. 266–268, July.
(19) General equation of the second degree [Set of formulas on a card]. Harvard University Press.

1896.

(20) On Cauchy's theorem concerning complex integrals. *Bulletin of the American Mathematical Society*, ser. 2, vol. 2, No. 5, pp. 146–149, Feb.
(21) Bessel's functions [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 2, No. 8, pp. 255–265, May.

(22) Linear differential equations and their applications. [Report by T. S. Fiske of a lecture at the Buffalo Colloquium]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 2, pp. 52–55, Nov.
(23) Heffter's Linear Differential Equations [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 2, pp. 86–92, Nov.
(24) Regular points of linear differential equations of the second order. Cambridge, Harvard University Press, 23 pp.

1897.

(25) Schlesinger's Linear Differential Equations [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 4, pp. 146–153, Jan.
(26) On certain methods of Sturm and their application to the roots of Bessel's functions. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 6, pp. 205–213, March.
(27) Review of Bailey and Woods: Plane and Solid Analytic Geometry. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 9, pp. 351–352, June.

1898.

(28) Examples of the construction of Riemann's surfaces for the inverse of rational functions by the method of conformal representation. By C. L. Bouton with an introduction by Maxime Bôcher. *Annals of Mathematics*, vol. 12, No. 1, pp. 1–26, Feb.
(29) The roots of polynomials which satisfy certain linear differential equations of the second order. *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 6, pp. 256–258, March.
(30) The theorems of oscillation of Sturm and Klein (first paper). *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 7, pp. 295–313, April.
(31) The theorems of oscillation of Sturm and Klein (second paper). *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 8, pp. 365–376, May.
(32) Note on some points in the theory of linear differential equations. *Annals of Mathematics*, vol. 12, No. 2, pp. 45–53, May.
(33) Note on Poisson's integral, *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 9, pp. 424–426, June.
(34) Niewenglowski's Geometry [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 9, pp. 448–452, June.
(35) The theorems of oscillation of Sturm and Klein (third paper). *Bulletin of the American Mathematical Society*, ser. 2, vol. 5, No. 1, pp. 22–43, Oct.

1899.

(36) Burkhardt's Theory of Functions [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 5, No. 4, pp. 181–185, Jan.
(37) On singular points of linear differential equations with real coefficients. *Bulletin of the American Mathematical Society*, ser. 2, vol. 5, No. 6, pp. 275–281, March.
(38) An elementary proof that Bessel's functions of the zeroth order have an infinite number of real roots. *Bulletin of the American Mathematical Society*, ser. 2, vol. 5, No. 8, pp. 385–388, May.
(39) Examples in the theory of functions. *Annals of Mathematics*, ser. 2, vol. 1, No. 1, pp. 37–40, Oct.

1900.

(40) On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 40–52, Jan.; also No. 4, p. 507, Oct.

(41) Some theorems concerning linear differential equations of the second order. *Bulletin of the American Mathematical Society*, ser. 2, vol. 6, No. 7, pp. 279–280, April.
(42) Application of a method of d'Alembert to the proof of Sturm's theorems of comparison. *Transactions of the American Mathematical Society*, vol. 1, No. 4, pp. 414–420, Oct.
(43) On linear dependence of functions of one variable. *Bulletin of the American Mathematical Society*, ser. 2, vol. 7, No. 3, pp. 120–121, Dec.
(44) Randwertaufgaben bei gewöhnlichen Differentialgleichungen. Encyklopädie der mathematischen Wissenschaften, II A 7a, pp. 437–463, Leipzig, Teubner.

1901.

(45) The theory of linear dependence. *Annals of Mathematics*, ser. 2, vol. 2, No. 2, pp. 81–96, Jan.
(46) Green's functions in space of one dimension. *Bulletin of the American Mathematical Society*, ser. 2, vol. 7, No. 7, pp. 297–299, April.
(47) Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence. *Transactions of the American Mathematical Society*, vol. 2, No. 2, pp. 139–149, April.
(48) An elementary proof of a theorem of Sturm. *Transactions of the American Mathematical Society*, vol. 2, No. 2, pp. 150–151, April.
(49) Non-oscillatory linear differential equations of the second order. *Bulletin of the American Mathematical Society*, ser. 2, vol. 7, No. 8, pp. 333–340, May.
(50) On certain pairs of transcendental functions whose roots separate each other. *Transactions of the American Mathematical Society*, vol. 2, No. 4, pp. 428–436, Oct.
(51) On Wronskians of functions of a real variable. *Bulletin of the American Mathematical Society*, ser. 2, vol. 8, No. 2, pp. 53–63, Nov.
(52) Picard's Traité d'Analyse [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 8, No. 3, pp. 124–128, Dec.

1902.

(53) Some applications of the method of abridged notation. *Annals of Mathematics*, ser. 2, vol. 3, No. 2, pp. 45–54, Jan.
(54) Review of Schlesinger: Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variabeln. *Bulletin of the American Mathematical Society*, ser. 2, vol. 8, No. 4, pp. 168–169, Jan.
(55) On the real solutions of two homogeneous linear differential equations of the first order. *Transactions of the American Mathematical Society*, vol. 3, No. 2, pp. 196–215, April.
(56) On systems of linear differential equations of the first order, *American Journal of Mathematics*, vol. 24, No. 4, pp. 311–318, Oct.
(57) Review of Gauss' Wissenschaftliches Tagebuch. *Bulletin of the American Mathematical Society*, ser. 2, vol. 9, No. 2, pp. 125–126, Nov.

1903.

(58) The Elements of Plane Analytic Geometry. By George R. Briggs. Revised and enlarged by Maxime Bôcher. New York, Wiley, 4 + 191 p.
(59) Singular points of functions which satisfy partial differential equations of the elliptic type. *Bulletin of the American Mathematical Society*, ser. 2, vol. 9, No. 9, pp. 455–465, June.

(60) On the uniformity of the convergence of certain absolutely convergent series. *Annals of Mathematics*, ser. 2, vol. 4, No. 4, pp. 159–160, July.

1904.

(61) Contribution to Sprechsaal für die Encyklopädie der Mathematischen Wissenschaften. *Archiv der Mathematik und Physik*, vol. 7, No. 3, p. 181, Feb.

(62) The fundamental conceptions and methods of mathematics. Address delivered before the Department of Mathematics of the International Congress of Arts and Science, St. Louis, Sept. 20, 1904. *Bulletin of the American Mathematical Society*, ser. 2, vol. 11, No. 3, pp. 115–135, Dec. Also in Congress of Arts and Science Universal Exposition, St. Louis, 1904, vol. 1. Boston, Houghton and Mifflin, 1905, pp. 456–473.

(63) A problem in statics and its relation to certain algebraic invariants. *Proceedings of the American Academy of Arts and Sciences*, vol. 40, No. 11, pp. 469–484, Dec.

1905.

(64) Linear differential equations with discontinuous coefficients. *Annals of Mathematics*, ser. 2, vol. 6, No. 3, pp. 97–111 (49–63), April.

(65) Sur les équations différentielles linéaires du second ordre à solution périodique. *Comptes Rendus de l'Académie des Sciences*, vol. 140, No. 14, pp. 928–931, April.

(66) A problem in analytic geometry with a moral. *Annals of Mathematics*, ser. 2, vol. 7, No. 1, pp. 44–48, Oct.

1906.

(67) Introduction to the theory of Fourier's series. *Annals of Mathematics*, vol. 7, No. 2, and No. 3, pp. 81–152, Jan. and April.

(68) On harmonic functions in two dimensions. *Proceedings of the American Academy of Arts and Sciences*, vol. 41, No. 26, pp. 577–583, March.

(69) Review of Picard: Sur le Développement de l'Analyse, etc. *Science*, n. s., vol. 23, No. 598, p. 912, June.

(70) Another proof of the theorem concerning artificial singularities. *Annals of Mathematics*, ser. 2, vol. 7, No. 4, pp. 163–164, July.

1907.

(71) Introduction to Higher Algebra. By Maxime Bôcher. Prepared for publication with the cooperation of E. P. R. Duval. New York, Macmillan, 11 + 321 pp.*

1908.

(72) Review of Bromwich: Quadratic Forms and their Classification by Means of Invariant Factors. *Bulletin of the American Mathematical Society*, ser. 2, vol. 14, No. 4, pp. 194–195, Jan.

(73) On the small forced vibrations of systems with one degree of freedom. *Annals of Mathematics*, ser. 2, vol. 10, No. 1, pp. 1–8, Oct.

1909.

(74) On the regions of convergence of power-series which represent two-dimensional harmonic functions. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 271–278, April.

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(75) Review of Runge: Analytische Geometrie der Ebene. *Bulletin of the American Mathematical Society*, ser. 2, vol. 16, No. 1, pp. 30–33, Oct.
(76) Review of d'Adhémar: Exercices et Leçons d'Analyse. *Bulletin of the American Mathematical Society*, ser. 2, vol. 16, No. 2, pp. 87–88, Nov.
(77) An introduction to the study of integral equations. Cambridge Tracts in Mathematics and Mathematical Physics, No. 10, Cambridge, England, University Press, 72 pp.*

1910.

(78) On semi-analytic functions of two variables. *Annals of Mathematics*, ser. 2, vol. 12, No. 1, pp. 18–26, Oct.
(79) Kowalewski's Determinants [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 18, No. 3, pp. 120–140, Dec.

1911.

(80) The published and unpublished work of Charles Sturm on algebraic and differential equations. Presidential address delivered before the American Mathematical Society, April 28, 1911. *Bulletin of the American Mathematical Society*, ser. 2, vol. 18, No. 1, pp. 1–18, Oct.
(81) Boundary problems and Green's functions for linear differential and difference equations. *Annals of Mathematics*, ser. 2, vol. 13, No. 2, pp. 71–88, Dec.
(82) Graduate work in mathematics in universities and in other institutions of like grade in the United States. General report. *United States Bureau of Education Bulletin*, No. 6, pp. 7–20. Also in *Bulletin of the American Mathematical Society*, ser. 2, vol. 18, No. 3, pp. 122–137, Dec.

1912.

(83) On linear equations with an infinite number of variables. By Maxime Bôcher and Louis Brand. *Annals of Mathematics*, ser. 2, vol. 13, No. 4, pp. 167–186, June.
(84) A simple proof of a fundamental theorem in the theory of integral equations. *Annals of Mathematics*, ser. 2, vol. 14, No. 2, pp. 84–85, Dec.

1913.

(85) Applications and generalizations of the conception of adjoint systems, *Transactions of the American Mathematical Society*, vol. 14, No. 4, pp. 403–420, Oct.
(86) Doctorates conferred by American universities [Letter to the editor], *Science*, n. s., vol. 38, No. 981, p. 546, Oct.
(87) Boundary problems in one dimension [A lecture delivered Aug. 27, 1912]. Proceedings of the Fifth International Congress of Mathematicians, Cambridge, England, University Press, vol. 1, pp. 163–195.

1914.

(88) The infinite regions of various geometries. *Bulletin of the American Mathematical Society*, ser. 2, vol. 20, No. 4, pp. 185–200, Jan.†
(89) On Gibbs's phenomenon. *Journal für die reine und angewandte Mathematik*, vol. 144, No. 1, pp. 41–47, Jan.

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* A second edition appeared in 1914.
† See vol. 22, (1915) No. 1, p. 40, Oct.

(90) Mathématiques et mathématiciens français. *Revue Internationale de l'Enseignement*, vol. 67, No. 1, pp. 20–31, Jan.
(91) Charles Sturm et les mathématiques modernes. *Revue du Mois*, vol. 17, No. 97, pp. 88–104, Jan.
(92) On a small variation which renders a linear differential system incompatible. *Bulletin of the American Mathematical Society*, ser. 2, vol. 21, No. 1, pp. 1–6, Oct.
(93) The smallest characteristic number in a certain exceptional case. *Bulletin of the American Mathematical Society*, ser. 2, vol. 21, No. 1, pp. 66–99, Oct.

1915.

(94) Trigonometry with the theory and use of logarithms. By Maxime Bôcher and H. D. Gaylord, New York, Holt, 9 + 142 pp.
(95) Plane analytic geometry with introductory chapters on the differential calculus. New York, Holt, 13 + 235 pp.

1916.

(96) Review of Gibb: A Course in Interpolation etc. and Carse and Shearer: A Course in Fourier's Analysis etc. *Bulletin of the American Mathematical Society*, ser. 2, vol. 22, No. 7, pp. 359–361, April.
(97) On the Wronskian test for linear dependence, *Annals of Mathematics*, ser. 2, vol. 17, No. 4, pp. 167–168, June.
(98) Syllabus of a Brief Course in Solid Analytic Geometry. Lancaster, New Era Press, 10 p.
(99) Leçons sur les méthodes de Sturm dans la théorie des équations différentielles linéaires et leurs développements modernes. Professées à la Sorbonne en 1913–14. Recueillies et rédigées par G. Julia. Paris, Gauthier-Villars, 6 + 118 pp.

1917.

(100) Note supplementary to the paper "On certain pairs of transcendental functions whose roots separate each other." *Transactions of the American Mathematical Society*, vol. 18, No. 4, pp. 519–521, Oct.

1918

(101) Concerning direction cosines and Hesse's normal form. *American Mathematical Monthly*, vol. 25, No. 7, pp. 308–310, Sept.

ON A THEOREM OF OSCILLATION.

BY PROFESSOR WILLIAM F. OSGOOD.

IN his book on series which represent the potential function Bôcher makes use of the first part of the following theorem.* All the variables are real.

THEOREM—*Let $\varphi(t)$ be continuous and monotonic in the interval $T \leqq t < \infty$, and let $\varphi(t)$ be always greater numerically than a certain positive constant γ:*

$$|\varphi(t)| > \gamma.$$

If $\varphi(t) < 0$, an arbitrary solution of the differential equation

$$\frac{d^2y}{dt^2} = \varphi(t)\, y,$$

oscillates an infinite number of times in any interval $T \leqq T' \leqq t < \infty$, in which it is considered, the amplitudes of the oscillations remaining finite.

If, furthermore, $\varphi(t)$ remains finite, the amplitudes of the oscillations do not become less than a certain positive constant. Moreover, the amplitudes vary monotonically, increasing when $\varphi(t)$ increases algebraically, and decreasing in the opposite case.

Bôcher states the theorem without the restriction that $\varphi(t)$ be monotonic, and outlines a suggestive dynamical proof. Professor Birkhoff's comment in the foregoing article led me to examine critically both theorem and proof. It is easy to see† that the theorem is not true under Bôcher's hypotheses. He needed the theorem, however, only in the restricted form above given. Moreover, he does not state the last paragraph of the theorem, this extension not being requisite for his purposes.

As regards this extension, it is not difficult to show that, if $\varphi(t)$ is not required to remain finite, the amplitudes of the oscillations may become infinitesimal, when t becomes infinite.

* Ueber die Reihenentwickelungen der Potentialtheorie; Teubner, 1894, p. 178.

† Cf. infra.

Proof. Let

$$p = p(t) = -\varphi(t), \qquad p_n = p(t_n).$$

To integrate the differential equation

$$\frac{d^2y}{dt^2} = -py, \tag{1}$$

proceed in the usual manner:

$$2\frac{dy}{dt}\frac{d^2y}{dt^2} = -2py\frac{dy}{dt},$$

$$\left(\frac{dy}{dt}\right)^2 = -2\int pydy + C.$$

We will now introduce with Bôcher the dynamical interpretation of the equation (1) as representing Newton's second law of motion for a particle of unit mass acted on by a central attractive force whose intensity, in absolute units, is py, and moving in a right line through the center of force O. Here, t represents the time, and y the distance, measured algebraically, of the particle from O.

We will determine the constant of integration for two particular cases.

Case A. The particle is projected from O in the sense of the increasing y with an initial velocity $v = c$. Then

$$\left(\frac{dy}{dt}\right)^2 = c^2 - 2\int_0^y pydy.$$

The amplitude h of this half-oscillation is given by putting $dy/dt = 0$, $y = h$. Hence

$$\int_0^h pydy = \frac{c^2}{2}. \tag{2}$$

Case B. The particle is released from rest at the point $y = h > 0$. Then

$$\left(\frac{dy}{dt}\right)^2 = 2\int_y^h pydy.$$

The velocity c with which it reaches the center of force is given by the equation

$$c^2 = 2\int_0^h pydy,$$

which is of the same form as (2).

Thus both cases lead to the same formal relation (2) between c and h.

Consider, now, an arbitrary solution y of (1) in the interval $T \leqq t' \leqq t < \infty$, and suppose

$$y|_{t=t'} > 0.$$

Since the force is attractive and its intensity is at least γy, the particle will be pulled in to the center of force, and the given solution will vanish at a definite instant* $t = t_0 > t'$.

It is sufficient, then, to consider a half-oscillation, i. e., an excursion from the center of force back to that point. Let the particle be projected at the instant $t = t_0$ from O with velocity v_0 in the direction of the positive y's.

I. Let the monotonic function $p(t)$ *decrease*. Then

$$p(t) > \gamma, \quad p(\infty) \geqq \gamma.$$

Corresponding to Case A, we have for the motion away from O to rest

$$\frac{p_1 h_1^2}{2} \leqq \uparrow \int_0^{h_1} pydy = \frac{v_0^2}{2}. \tag{3}$$

Hence

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* The argument can be equally well carried through analytically. The curve corresponding to the solution in question, $y = f(t)$, is concave downward, since $d^2y/dt^2 < 0$. If $f'(t') > 0$, the curve must have a maximum and return to the initial level, $y = f(t') = b$. For, while it is above that level,

$$d^2y/dt^2 = -py < -\gamma b = -k.$$

Consider the parabola for which $d^2y/dt^2 = -k$, and which is tangent initially to the above curve:

$$Y = b + f'(t')(t - t') - \tfrac{1}{2}k(t - t')^2. \tag{i}$$

Then the given curve will lie below the parabola so long as it lies above the line $y = b$. For, by Taylor's theorem with the remainder,

$$y = b + f'(t')(t - t') + \tfrac{1}{2}f''(\tau)(t - t')^2, \quad t' < \tau < t. \tag{ii}$$

Hence

$$Y - y = \tfrac{1}{2}[py - k]_{t=\tau}\,(t - t')^2,$$

and the bracket is positive so long as $y \geqq b$.

Thus the curve $y = f(t)$ cuts the line $y = b$ a second time, and at this point the slope is negative. From now on the curve lies below its tangent at this point, till it cuts the axis of t.

$$h_1 \leqq \frac{v_0}{\sqrt{p_1}} \leqq \frac{v_0}{\sqrt{\gamma}}.$$

Corresponding to Case B, we have for the motion back* to O

$$\frac{p_1 h_1^2}{2} \geqq \downarrow \int_0^{h_1} p y dy = \frac{v_2^2}{2}. \tag{4}$$

From (3) and (4) we infer that

$$v_2 \leqq v_0.$$

It thus appears that, as the motion continues, the successive values $v_0, v_2, v_4, \cdots$ never increase, and hence h_n remains finite, since

$$h_{n+1} \leqq \frac{v_{2n}}{\sqrt{\gamma}} \leqq \frac{v_0}{\sqrt{\gamma}}.$$

To show that h_n remains greater than a certain positive constant, we replace (3) and (4) by the new relations

$$\frac{p_0 h_1^2}{2} \geqq \uparrow \int_0^{h_1} p y dy = \frac{v_2^2}{2}; \tag{5}$$

$$\frac{p_2 h_1^2}{2} \leqq \downarrow \int_0^{h_1} p y dy = \frac{v_2^2}{2}. \tag{6}$$

Hence,

$$h_1 \geqq \frac{v_0}{\sqrt{p_0}}, \qquad \frac{v_2}{\sqrt{p_2}} \geqq \frac{v_0}{\sqrt{p_0}}.$$

On writing down the corresponding relations for the further half-oscillations and taking v_{2m} always to mean the numerical value of dy/dt at O,

$$\frac{v_4}{\sqrt{p_4}} \geqq \frac{v_2}{\sqrt{p_2}},$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{v_{2n}}{\sqrt{p_{2n}}} \geqq \frac{v_{2n-2}}{\sqrt{p_{2n-2}}},$$

we obtain

$$h_{n+1} \geqq \frac{v_{2n}}{\sqrt{p_{2n}}} \geqq \frac{v_0}{\sqrt{p_0}},$$

* p, considered as a function of y, is multiple-valued; but in each of the cases A and B we are dealing only with a single-valued branch of the function.

and thus the theorem is proven for the case of a decreasing function $p(t)$, the corresponding function $\varphi(t)$ increasing.

Finally, h_n increases monotonically,

$$h_{n+1} \geqq h_n,$$

since from (5) and (6)

$$\frac{v_0}{\sqrt{p_0}} \leqq h_1 \leqq \frac{v_2}{\sqrt{p_2}},$$

etc.

II. When $p(t)$ is an *increasing* function, the proof is given in a similar manner. Instead of (3) and (4) we now have

$$\frac{p_0 h_1^2}{2} \leqq \int_0^{h_1} pydy = \frac{v_0^2}{2}; \tag{7}$$

$$\frac{p_2 h_1^2}{2} \geqq \int_0^{h_1} pydy = \frac{v_2^2}{2}. \tag{8}$$

Here, we have

$$h_1 \leqq \frac{v_0}{\sqrt{p_0}}, \qquad \frac{v_2}{\sqrt{p_2}} \leqq \frac{v_0}{\sqrt{p_0}}.$$

Hence

$$h_{n+1} \leqq \frac{v_{2n}}{\sqrt{p_{2n}}} \leqq \frac{v_0}{\sqrt{p_0}},$$

and h_n remains finite.

To show that h_n does not fall below a positive constant, write

$$\frac{p_1 h_1^2}{2} \geqq \int_0^{h_1} pydy = \frac{v_0^2}{2}; \tag{9}$$

$$\frac{p_1 h_1^2}{2} \leqq \int_0^{h_1} pydy = \frac{v_2^2}{2}. \tag{10}$$

Hence

$$h_1 \geqq \frac{v_0}{\sqrt{p_1}} \geqq \frac{v_0}{\sqrt{\gamma}}, \qquad v_2 \geqq v_0,$$

and thus

$$h_{n+1} \geqq \frac{v_0}{\sqrt{\gamma}}.$$

From (7) and (8) it follows that

$$\frac{v_2}{\sqrt{p_2}} \leqq h_1 \leqq \frac{v_0}{\sqrt{p_0}},$$

etc.

Hence

$$h_{n+1} \leqq h_n,$$

and h_n decreases monotonically.

Physical Interpretation.—A physical interpretation of the problem just discussed is given by the small oscillations of a simple pendulum, the cord of which, in Case I, is gradually lengthened, monotonically, remaining finite; while in Case II the cord is monotonically shortened, never becoming less than a positive fixed length.

The same physical picture is useful, too, for showing that the theorem, stated without restriction on $\varphi(t)$, is false. For the way a child swings higher and higher is to lengthen the equivalent simple pendulum on the downward arc, and suddenly to shorten it as the upward arc begins.

Bôcher's theorem (*b*), l. c., namely that, when $\varphi(t)$ is continuous and

$$0 < g \leqq \varphi(t) \leqq G,$$

there is one and only one solution which remains finite, and this vanishes at infinity, is correct without any further restriction on the function φ. An elementary analytic proof can be given by the aid of the law of the mean and Taylor's theorem with the remainder, carried through the term of the second order.

HARVARD UNIVERSITY,
CAMBRIDGE, MASSACHUSETTS,
December 8, 1918.

PROOF OF A PROPERTY OF THE NORM OF A CYCLOTOMIC INTEGER.

BY MR. H. S. VANDIVER.

(Read before the American Mathematical Society April 27, 1918.)

KUMMER in his first proof of the general law of reciprocity between two ideals in a regular cyclotomic algebraic field gave a theorem* which forms a link in his chain of reasoning. Let

* *Abhandlungen Berlin Academy*, 1859, p. 119, formula (7).

$$\omega(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{p-2} x^{p-2},$$
$$\omega = a_0 + a_1 \alpha + \cdots + a_{p-2} \alpha^{p-2},$$

where $a_0, a_1, \cdots, a_{p-2}$ are integers, p is a prime, $\alpha = e^{2i\pi/p}$ and ω is prime to p. Write

$$D_s(\omega) = \left[\frac{d^s \log \omega(e^v)}{dv^s}\right]_{v=0},$$

and let $N(\omega)$ be the norm of ω. Then we shall prove that

$$(1) \qquad D_{p-1}(\omega) \equiv \frac{\omega(1)^{p-1} - N(\omega)}{p} \pmod{p},$$

which result was given by Kummer in the article cited. Kummer's proof depended on certain expansions in logarithmic series. In the present paper we proceed from another point of view. Put

$$N(\omega) = \omega(\alpha)\omega(\alpha^2)\cdots\omega(\alpha^{p-1}).$$

In this relation α may be replaced by an indeterminate x if we add a suitable multiple of $x^{p-1} + x^{p-2} + \cdots + 1$. Letting $x = e^v$, we may then write

$$(2) \qquad \omega(e^v)\omega(e^{2v})\cdots\omega(e^{(p-1)v}) = N(\omega) + \frac{e^{vp} - 1}{e^v - 1} X,$$

where X is a rational integral function of e^v. Put $v = 0$ in this relation; we obtain

$$(3) \qquad pX(1) + N(\omega) = (\omega(1))^{p-1}.$$

Taking logarithms in (2), differentiating $(p-1)$ times with respect to v, and setting $v = 0$ we have, if $V = (e^{vp} - 1)/(e^v - 1)$,

$$(1 + 2^{p-1} + 3^{p-1} + \cdots + (p-1)^{p-1}) D_{p-1}(\omega)$$
$$(4) \qquad \equiv \left[\frac{d^{p-2}}{dv^{p-2}}\left(\frac{1}{N + VX}\frac{d(VX)}{dv}\right)\right]_{v=0}$$
$$\equiv \left[\frac{1}{N + VX}\frac{d^{p-1}(VX)}{dv^{p-1}}\right]_{v=0} \pmod{p},$$

since $D_s(V) \equiv 0 \pmod{p}$ for $S < p - 1$ and

$$[N + VX]_{v=0} \not\equiv 0 \pmod{p},$$

ω and therefore N being prime to p. We have further

$$(5) \quad \left[\frac{d^{p-1}(VX)}{dv}\right]_{v=0} \equiv X(1)\left[\frac{d^{p-1}V}{dv}\right]_{v=0} \equiv -X(1) \pmod{p}.$$

Comparison of (5), (4) and (3) gives the theorem (1).

PHILADELPHIA,
April, 1918.

TRAJECTORIES AND FLAT POINTS ON RULED SURFACES.

BY MR. J. K. WHITTEMORE.

(Read before the American Mathematical Society April 28 and October 27, 1917.)

§1. *Introduction.* In the following paper we determine the flat points* of a ruled surface with real rulings, and prove a new property of the orthogonal trajectories of the rulings. This property may be extended to any isogonal trajectory of the rulings, not itself a ruling, and may be regarded as a generalization of Bonnet's familiar theorem.†

Let S be a ruled surface with real rulings, g any such ruling, C an orthogonal trajectory of the rulings, generally not a straight line; let the coordinates of any point of C be x_0, y_0, z_0, and consider these as functions of v, the arc of C. When C is not a straight line let the direction cosines of its tangent be α, β, γ, of its principal normal be l, m, n, and of its binormal be λ, μ, ν; let R and T be respectively the radii of curvature and torsion of C, ψ the angle measured from the principal normal towards the binormal to the direction chosen as positive on g. We suppose C to be a rectifiable curve generally without singular points in the portion considered; the curvature $1/R$ and the torsion $1/T$ shall have finite first derivatives with respect to v, and ψ shall have a finite second derivative.

The surface S is given by

$$(1) \qquad x = x_0 + uL, \qquad L = l\cos\psi + \lambda\sin\psi,$$

* Flat points are defined in § 4 of this paper.
† See Eisenhart, Differential Geometry, p. 248.

with similar equations for y and z, where u is the length measured from C on g in the positive direction. If C is a straight line S is a right conoid. To give its equations we choose C as the X axis and let ψ be the angle made by the positive direction on g measured from the Y axis towards the Z axis. We have

$$(1') \qquad x = v, \qquad y = u\cos\psi, \qquad z = u\sin\psi.$$

The linear element of (1) is given by

$$(2) \qquad ds^2 = du^2 + [1 - 2uq + u^2(p^2 + q^2)]dv^2$$

$$p = \frac{1}{T} - \psi' \qquad q = \frac{\cos\psi}{R}.$$

For (1') ds is also given by (2) making $1/R = 1/T = 0$. S is developable if p is identically zero; if q is also identically zero S is a plane or a cylinder.

It is well known* that the linear element of S may be given the form

$$(3) \qquad ds^2 = du_1^2 + [(u_1 - a)^2 + b^2]dv_1^2,$$

where the curves (v_1), that is v_1 constant, are the rulings, u_1 is the length measured along a ruling from an orthogonal trajectory of the rulings, $u_1 = 0$, a depending on v_1 alone the distance from this trajectory to the central point of the ruling so that $u_1 = a$ is the equation of s, the line of striction of S; b also depending on v_1 alone is the parameter of distribution of S. The total curvature K of S from (2) and (3) is

$$K = \frac{-p^2}{[1 - 2uq + u^2(p^2 + q^2)]^2} = \frac{-b^2}{[(u_1 - a)^2 + b^2]^2}.$$

Comparing (2) and (3), we have

$$a = \frac{q}{p^2 + q^2} \qquad b = \frac{p}{p^2 + q^2},$$

the sign of b being determined in the usual way.†

§ 2. *A property of the trajectories.* Suppose a point v of C is on s; for this point $a = 0$, and $q = \cos\psi/R = 0$. If $1/R = 0$ for all points of C that curve is a straight line, a

* Eisenhart, p. 247.
† Eisenhart, p. 246.

geodesic, and by Bonnet's theorem is s; if $1/R = 0$ for the point v, but not for all points of C, it follows that C is at this point tangent to the asymptotic line of S, not g, through v, unless the point v is a flat point. If $1/R \neq 0$, $\cos\psi = 0$ and the normal to S at v coincides with the principal normal of C, and the geodesic curvature of C vanishes at the point. If at a point of intersection of C and s the former is tangent to an asymptotic line its principal normal is not necessarily normal to S, as may be seen from the following construction: let C be any curve satisfying the conditions of § 1, v a point of C for which $1/R = 0$, $1/T \neq 0$, let S be a surface of normals to C, ψ having any *constant* value. The point v is on s since $q = 0$, $p = 1/T \neq 0$; S has at this point an ordinary point which is not a flat point since $K \neq 0$. The curve C is tangent to an asymptotic line at v, and the angle of the normal to S and the principal normal of C at v is the complement of ψ.

Suppose conversely that v is a point of C where the geodesic curvature vanishes; then $\cos\psi = 0$, $q = 0$, and, unless $p = 0$, $a = 0$, and v is on s. If $p = 0$, $K = 0$ for v and consequently for all points of the ruling through v. Such a ruling we shall call a *zero ruling*. In the next section we examine the behavior of orthogonal trajectories of the rulings at points of intersection with zero rulings.

The properties of orthogonal trajectories just proved may be extended without modification to all isogonal trajectories—a ruling itself not being considered, however, as a trajectory. We may state the results of this section in

Theorem 1: At a point of intersection of an isogonal trajectory of the rulings of a surface S with the line of striction the geodesic curvature of the trajectory vanishes unless the tangent to the latter is asymptotic or the point is a flat point of S; conversely, a point of a trajectory where the geodesic curvature vanishes is either on the line of striction or on a zero ruling.

§ 3. *Trajectories on zero rulings.* If the total curvature K vanishes for v and a finite value of u, $p = 0$; either $b = 0$, or a or b or both become infinite. If $q \neq 0$, $b = 0$ and $a = 1/q$ is finite; if $q = 0$ either a or b becomes infinite. We distinguish two cases of zero rulings: (I) $p = 0$, $q \neq 0$; (II) $p = q = 0$. That this is a distinction independent of the trajectory C appears from the corresponding properties of a and b, as also from properties of the point of the spherical indicatrix of S corresponding to the ruling, which are stated below.

Let D, D', D'' be the coefficients of the second fundamental form of S; D is identically zero, and $K = 0$ gives $D' = 0$. The last condition gives

$$|LL'\alpha| = 0 \qquad L' = -\alpha q + (l \sin\psi - \lambda \cos\psi)p.$$

For (I)

$$\frac{\alpha}{L'} = \frac{\beta}{M'} = \frac{\gamma}{N'} = -\frac{1}{q}.$$

For (II)

$$L' = M' = N' = 0.$$

In (I) the point of the spherical indicatrix of S corresponding to v, that is the point of coordinates (L, M, N), is a simple point of the indicatrix, and the tangent to the indicatrix at v is parallel to the tangent to C at v. The orthogonal trajectory intersecting a zero ruling (I) on s has at the point of intersection a singular point, as may be seen in various ways: (1) $a = 0$ gives $R = 0$; (2) the tangent to the spherical indicatrix at a point corresponding to a ruling g is parallel to the normal to S at the point of intersection of g and s,* but we have seen that the tangent to the indicatrix is parallel to the tangent to the orthogonal trajectory, a contradiction in the case of an ordinary point of the trajectory on s. This remark proves incidentally that the tangent to any other orthogonal trajectory at a point of intersection with a zero ruling (I) is parallel to the normal to S at the intersection of the ruling with s. Since in (I) $\cos\psi \neq 0$ no orthogonal trajectory has zero geodesic curvature at a point on the ruling.

In (II) $q = 0$ gives $\cos\psi = 0$ or $1/R = 0$; if $\cos\psi = 0$ the trajectory has zero geodesic curvature; if $1/R = 0$ the point is a flat point of S, for the ruling has the only asymptotic direction at a point for which $K = 0$. The behavior of a trajectory at a flat point will be considered in the following section. To a zero ruling (II) corresponds a singular point of the spherical indicatrix since

$$L' = M' = N' = 0.$$

An example of a zero ruling (II) is furnished by the surface of binormals of a curve C at a point for which the torsion vanishes. In this example b is infinite and a may have any value.

* Eisenhart, p. 351. This does not apply to a developable surface.

§ 4. *Flat points.* A flat point of a surface is a point where the curvature of every normal section is zero; it may also be defined as an umbilic where the total curvature of the surface vanishes.* The conditions for an umbilic are, in the usual notation,

$$\frac{D}{E} = \frac{D'}{F} = \frac{D''}{G};$$

for a flat point $D = D' = D'' = 0$. For S, with the coordinates u, v used in the preceding sections, $D = 0$, $E = 1$, so that such a surface has no umbilics other than flat points. It is worth while at this point to emphasize the requirement that S be a surface with *real* rulings, for there exists a real ruled surface with no flat points all of whose real points are umbilics—the sphere.

To determine the flat points of a surface S we have, in addition to the identity $D = 0$ and the condition for zero total curvature $D' = 0$, from which v may be found, the equation for u, $D'' = 0$, which gives

$$(4) \quad u^2|LM'N''| + u\{|x_0''MN'| - |x_0'MN''|\} + |x_0''Mz_0'| = 0,$$

in which each bracket of three letters represents a determinant with principal diagonal in evidence.

For a zero ruling (I) equation (4) has two roots,

$$u = -\frac{1}{\sqrt{\Sigma L'^2}} = \frac{1}{q} \qquad u = -\frac{|x_0''MN'|}{|LM'N''|} = \frac{\sin\psi}{Rt}$$

$$t = \frac{\sin\psi\cos\psi}{R^2} - p'.$$

The first value gives the central point of the ruling which is generally not a flat point but is a singular point of the coordinate curve (u), the orthogonal trajectory through the point. The second value gives the same point if $p' = 0$, $\sin\psi \neq 0$; if $p' = \sin\psi = 0$, $D'' = 0$ for all u's, and all points of the ruling are flat points; if $p' \neq 0$, $t \neq 0$, the general case, there is one flat point; if $p' \neq 0$, $t = 0$ there is no flat point.

For a zero ruling (II) the coefficient of u^2 in (4) vanishes, and that equation gives for a flat point one value,

* Scheffers, Theorie der Flächen, 2d ed., pp. 126, 146.

$$u = \frac{|x_0'' M z_0'|}{|x_0' M N''|} = -\frac{\sin\psi}{Rp'}.$$

The ruling has one finite flat point if $p' \neq 0$; if $p' = 0$ there is no flat point unless also $1/R = 0$; if $p' = 1/R = 0$ all points of the ruling are flat points for $D'' = 0$ for all u's. The case of the right conoid, where C is a straight line, is included in (II), for since $1/R = 0$, $q = 0$; one flat point is on C, and all other or no other points of the ruling are flat points according as $p' = -\psi''$ does or does not vanish.

In (I) we have $a\cos\psi = R$, so that the projection of the central point of the ruling on the osculating plane of any orthogonal trajectory C, satisfying the requirements of § 1, is the center of curvature of the corresponding point of the trajectory. In the case of a single flat point in (I) suppose for the moment C is the orthogonal trajectory through that point and that C satisfies the requirements of § 1; we have

$$u = \frac{\sin\psi}{Rt} = 0.$$

Since a is finite $1/R \neq 0$, and $\sin\psi = 0$, and the ruling is the principal normal of the trajectory through the flat point, and the central point of the ruling is the corresponding center of curvature of that trjaectory. If in (I) all points of the ruling are flat points $\sin\psi = 0$ for any orthogonal trajectory satisfying the requirements of § 1; the ruling is the principal normal and the central point the center of curvature of all such trajectories. The latter case is illustrated by taking for S a plane regarded as the locus of tangents of one of its curves s. It may be proved if $p' = 0$, $\sin\psi \neq 0$, no flat point, that the principal normals of all orthogonal trajectories at points of the zero ruling are parallel. This case is illustrated on any ruling of a developable surface whose points are not flat points. In the case remaining in (I), $p' \neq 0$, $t = 0$, no flat point, there is no simple relation between the ruling and the trajectories.

In (II) the ruling is the binormal of every orthogonal trajectory not intersecting the ruling at a flat point. If there is a single flat point on the ruling, ψ may have any value at this point; if all points of the ruling are flat points there is no simple relation between the ruling and the trajectories.

By a discussion similar to the preceding we may prove analogous properties of isogonal—not orthogonal—trajectories of the rulings of S. We choose such a trajectory as the direc-

trix C, and suppose C cuts the rulings at the angle $A \neq 0$. No trajectory has zero geodesic curvature at a point of intersection with a zero ruling (I). Every trajectory through the central point of a ruling (I) has at this point a singular point, but if $A \neq \pi/2$ the coordinate curve (u) through the central point has not a singular point at this point, so that the point is generally not a singular point of the surface. This case is again illustrated by a plane regarded as the locus of the tangents of one of its curves. Every trajectory intersecting a zero ruling (II) at a point not a flat point has zero geodesic curvature at this point.

We consider finally the behavior of the flat points when a surface S is deformed into a surface of the same sort with correspondence of rulings.* From (2) it follows that in such deformation the values of p^2 and q are unchanged for any point of C; therefore a zero ruling remains such and moreover remains always in the same case, (I) or (II). Since the arc v is also unchanged the same is true of the absolute value of p', but $1/R$ and ψ are generally changed, so that the flat point moves on the ruling during deformation. In (I) deformation cannot change a ruling of one flat point into a ruling of all flat points, but can change either of these into a ruling with no flat point. In (II) a ruling of one flat point remains such during deformation, but a ruling of no flat points may be changed into a ruling of all flat points. A developable surface never has a ruling with a single flat point. The deformations for rulings of all and no flat points are illustrated for (I) by a plane regarded as the locus of tangents of one of its curves, for (II) by a cylinder. If all points of a surface are flat points the surface is a plane.† We remark that all cases considered in this section actually occur, for a surface S may be constructed to satisfy the requirements of § 1 with values arbitrarily assigned to $1/R$, ψ, p, p' at a point of C.

We state the chief results of this section in

Theorem 2: On a zero ruling of a surface S there is generally one flat point. The geodesic curvature of an isogonal trajectory of the rulings at a point of intersection with a zero ruling, not a flat point, is zero for all trajectories or for none.

YALE UNIVERSITY, 1918.

* If two ruled surfaces are applicable the rulings correspond necessarily unless both surfaces are applicable to the same quadric. See Eisenhart, p. 347.

† Scheffers, p. 146.

POINTS OF VIEW OF CAUCHY AND WEIERSTRASS IN THE THEORY OF FUNCTIONS.

Leçons sur les Fonctions monogènes uniformes d'une Variable complexe. Par EMILE BOREL. Rédigées par GASTON JULIA. Gauthier-Villars, Paris, 1917. xii + 165 pp.

THE point of departure in this monograph is the opposition between the points of view of Cauchy and Weierstrass in the definition of functions of a complex variable called *monogenic* by Cauchy and *analytic* by Weierstrass. The conclusion which is reached—and it is supported by detailed investigation and significant results—is that the point of view of Cauchy is the more fundamental one. In fact, it is contended that the work of Weierstrass is confined to a class of functions more special than that which should be considered and that in a more general class the central theorems are maintained, so that there is no reason for confining attention entirely to the analytic functions of Weierstrass.

The researches of which we have here an exposition convenient for the reader go back as far as a quarter century. Borel entered upon some of the matters involved as early as 1892 and published the first results in 1894. But it was as late as 1912 when he made known the principal results of the theory of non-analytic monogenic functions. This was in his lecture before the Fifth International Congress of Mathematicians at Cambridge. This earlier exposition was incomplete in many respects; and it was only in the course of lectures at Paris of which this book is the reproduction that he has completely developed the theory, entering into detail in the demonstrations and leaving no delicate point without careful illumination.

It was intended that this book should appear at the end of 1914; but M. Gaston Julia, a student at the Ecole Normale Supérieure, who was charged with preparing the notes for publication, was called into the army and was wounded in January, 1915. Notwithstanding his sufferings, he continued with the labor of seeing the manuscript through the press even though he was also at the same time engaged in remarkable researches of his own suitable "for perpetuating French mathematical traditions." The difficulties under which the

proofreading was done probably account for the rather large number of minor misprints, more than a score of which the reviewer detected. None of them is such as to cause the reader serious inconvenience.

The intimate connection between the theory of functions of a complex variable and the theory of the potential in mathematical physics has been an important guide to Borel in his researches on non-analytic monogenic functions, as he has pointed out in his Cambridge lecture; he has been helped particularly by the analogies of the new theory with the theories of molecular physics so much in consideration among scientists in very recent years. The non-analytic monogenic functions correspond to the case where the singular regions are at the same time extremely small and extremely numerous.

In the monograph under consideration attention has been confined to what may be called the purely mathematical theory of monogenic functions, and none of the physical applications just indicated are developed. In his preface the author states that it had originally been his intention to publish simultaneously with this monograph another in which were to be given the applications of these theories to the problems of molecular physics. But he finds himself compelled to leave off this study "until the moment when the victory of civilization . . . over a brutal aggression shall give us the right to take up again the thoughts and labors of times of peace." Some indication of what he has in mind in this further study is to be found in the second note in the monograph under review and in the seventh note in his Introduction géométrique à quelques Théories physiques. Perhaps one must wait until the appearance of a completed exposition of these applications before reaching a final judgment on the matters insisted upon by Borel; but the development already before us seems to indicate that the point of view of Cauchy is indeed more fundamental than that of Weierstrass and in a way which will require our thinking of the latter simply as the most important and most elegant case of the former.

Cauchy has laid down the conditions of monogeneity which lie at the base of the theory of functions of a complex variable and in his integral has created a matchless instrument of demonstration for dealing with this theory. In the study of differential equations he employed the process of analytic continuation. The rôle of Weierstrass was to make more pre-

cise certain points of the latter theory, notably that which pertains to the form of the domains. We may speak of the domains of Weierstrass, or the domains W, to mean those domains in which an analytic function may be defined by the process of analytic continuation as employed by Weierstrass. These domains may be defined geometrically and their properties be developed without reference to functions which are or may be defined in them.

The first chapter of this monograph (pages 1–23) is devoted to certain generalities (pages 1–5), the definition of a monogenic function (pages 5–10), geometric properties of domains W (pages 10–18), and the representation of an analytic function defined in a domain W (pages 19–23), the latter being given in the form of a Cauchy integral with the consequent series of Taylor and Laurent. Thus the chapter deals throughout with classic matter; the discussion of domains W may be singled out as particularly pleasing.

Chapter II (pages 24–54) is devoted to an application of the integral of Cauchy to the development in a series of polynomials of a function defined in a domain of Weierstrass. The first fourteen pages are given to an exposition of the method of Runge and the remainder of the chapter to applications to certain particular developments.

The third chapter (pages 55–72) is devoted to certain remarkable consequences of the development in series of polynomials and to an extension of the theory of analytic continuation. Here we have some preliminary aspects of Borel's reasons for insisting upon the inadequacy of the Weierstrass theory from the point of view of the more general theory of monogenic functions, the arguments here being presented in such way as to enable one to pass to the conclusion as directly as possible from first considerations. Then, after a necessary preliminary treatment of sets of points of measure zero (in Chapter IV, pages 73–124), he introduces in Chapter V (pages 125–150) the domains C, or domains of Cauchy, and develops the first elements of the general theory of non-analytic monogenic functions defined in these domains. There is nothing to indicate that these are the most general domains in which monogenic functions may be defined; but they are more general than the domains of Weierstrass.

Poincaré, with great ingenuity, constructed analytic expressions presenting certain remarkable singularities and be-

lieved that he had sufficient grounds for concluding to the impossibility of extending the theory of analytic functions beyond the bounds fixed by Weierstrass; that is, he believed in the impossibility of a process of analytic continuation by which one could extend a function beyond the domain of Weierstrass. With this negative result Borel has been for a long time in disagreement, at first thinking that it was not established and now finally justifying his position by means of the detailed results set forth in this monograph. Particularly, he has objected to the conclusion that the domain throughout which a function could be continued analytically by means of power series necessarily constitutes its *natural* domain of existence, that the function does not exist outside of this domain, and that every attempt to pierce this Unknown is destined to failure and contradiction. In this connection it is interesting to note the following from the preface to this monograph:

"A la suite des travaux de Poincaré que j'ai rappelés il y un instant, ce point de vue paraissait universellement admis; mais tandis que Poincaré accueillait avec bienveillance le premier essai dans lequel je montrais que les limites imposées par Weierstrass n'étaient pas aussi infranchissables qu'on l'avait cru, les disciples fidèles de Weierstrass ne consentaient même pas à discuter; je me rappellerai toujours l'étonnement avec lequel je vis M. Mittag-Leffler, auquel j'avais essayé d'exposer mes projets de recherches, ne faire aucun effort pour entrer dans ma pensée et se contenter de retirer de sa malle un Mémoire de Weierstrass pour me montrer une phrase qui devait clore définitivement toute discussion: Magister dixit."

In Chapter III we have an interesting example bearing on this matter of dispute and one which seems to settle the question definitively in favor of Borel's contention. Let a_1, a_2, a_3, $\cdots$ be the rational numbers between 0 and 1 arranged in enumerable sequence and consider the infinite series

$$(1) \qquad f(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - e^{2\pi i a_n}}, \qquad i = \sqrt{-1},$$

where the A's are constants such that the series $A_1 + A_2 + \cdots$ converges absolutely. It is easy to see that this series defines a function which is holomorphic at every point in the interior of the unit circle about the point zero. It may be shown that this circle is a *natural* boundary, in the sense of

Weierstrass, of the function defined by the series at an interior point of this circle. Likewise the series defines a function which is holomorphic at every point exterior to this unit circle and the circle is a natural boundary of the function thus defined. In the theory of Weierstrass there is no means of continuation by which one may establish a connection between the function defined by (1) in the interior of the unit circle and that so defined in the exterior of this circle. In fact, from the point of view of Weierstrass they are to be treated as unrelated functions.

Nevertheless Borel shows that a suitable continuation does exist for establishing the connection between these functions, this being constituted by a series of polynomials. In fact, one can form a series of polynomials

$$\sum_{p=0}^{\infty} P_p(z)$$

which converges uniformly on every finite segment of every straight line of argument $m\sqrt{2}+n$, where m and n are integers and $m \neq 0$, an infinity of which lines lie in every angle issuing from the origin; on each of these lines the series converges to the sum of the series in (1).

R. D. CARMICHAEL.

SHORTER NOTICE.

Elements of Optics for the Use of Schools and Colleges. By GEORGE W. PARKER, M. A. London, New York, and Bombay, Longmans, Green and Company, 1915. vi + 122 pp.

THE mathematical prerequisites necessary for reading this little book have been reduced to a minimum. A student whose knowledge of mathematics is limited to an acquaintance with elementary geometry, the solution of simple algebraic equations and a few fundamental propositions in trigonometry will be able to follow the treatment at all places. The knowledge of physical phenomena presupposed is also reduced to an extreme minimum. The book is therefore of a strictly elementary character. It is written in a satisfactory style and its material is arranged in interesting sequence, so that it may be

recommended to one who seeks pleasing applications of the most elementary mathematics to a chapter in scientific theory.

The chief merits of the exposition as an elementary treatment of its subject matter are intimately dependent upon the straightforward and simple manner of presentation on account of which the reader is able to follow the development with striking unity of effort and with little loss of energy consumed through divergent operations of thought. This renders the book particularly valuable for the learner who needs to concentrate attention upon the main issues in order to understand them thoroughly.

The effort to attain the advantages just mentioned has also led the author into the chief defects of his exposition. These are associated with the description of a special case as though it were the general case. Thus a lens is defined (page 56) as "a transparent body bounded by two spherical surfaces" and the student is left without any hint that lenses may also be of other forms. The most usual form of the kaleidoscope is described (page 13) as if there were no other form. A similar defect is in such a definition as that of optics (page 1) as "the science which treats of the properties of this mysterious agent" light, whereas the book itself deals with only a very narrow range of the properties of light and the student is given no hint of the more fundamental matters not treated in the book. The mathematical reader also feels a certain uneasiness in the free use of "infinity" (as on pages 84, 89, 113, and elsewhere) and in the uncritical use of processes of approximation. Nevertheless these minor defects do not obscure the real interest and value of this very elementary exposition.

R. D. CARMICHAEL.

NOTES.

THE fourth annual meeting of the Mathematical Association of America was held at the University of Chicago on Friday, December 27, 1918, in connection with the annual meeting of the American Mathematical Society. The morning programme included a conference on "Deductions from war time experiences with respect to the teaching of mathematics," a paper on "An experiment in supervised study," by D. R.

CURTISS, and the election of officers. At the afternoon joint session with the American Mathematical Society the following papers were presented: "Some mathematical features of ballistics," by A. A. BENNETT; "How the map problem was met in the war," by KURT LAVES; "Notes concerning recent books on navigation," by ALICE B. GOULD; "Statistics methods for preparation for war department service," by H. L. RIETZ; "Ordnance problems," by W. D. MACMILLAN; "Practical exterior ballistics," by P. L. ALGER; "The effect of the earth's rotation and curvature on the path of a projectile," by W. H. ROEVER; "On low velocity high angle fire," by H. F. BLICHFELDT. The evening was devoted to a joint dinner of the two organizations at the Quadrangle Club.

THE December number (volume 20, number 2) of the *Annals of Mathematics* contains the following papers: "The gamma function in the integral calculus (concluded)," by T. H. GRONWALL; "Invariants which are functions of parameters of the transformation," by O. E. GLENN; "A theorem on exhaustible sets connected with developments of positive real numbers," by HENRY BLUMBERG; "Solution of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ and its application to some geometrical problems," by ALEXANDER PELL; "A general method of summation of divergent series," by L. L. SMAIL.

THE fourth volume (1918) of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers:—number 5 (May): "The structure of an electromagnetic field," by H. BATEMAN; "Invariants which are functions of parameters of the transformation," by O. E. GLENN; number 7 (July): "On the representation of a number as the sum of any number of squares, and in particular of five or seven," by G. H. HARDY; number 8 (August): "Arithmetical theory of certain Hurwitzian continued fractions," by D. N. LEHMER; "On closed curves described by a spherical pendulum," by ARNOLD EMCH; number 9 (September): "On the α-holomorphisms of a group," by G. A. MILLER; number 10 (October) "Invariants and canonical forms," by E. J. WILCZYNSKI; number 11 (November): "On certain projective generalizations of metric theorems, and the curves of Darboux and Segre," by G. M. GREEN; number 12 (December): "On Jacobi's extension of the continued fraction algorithm," by

D. N. LEHMER; "A characterization of Jordan regions by properties having no reference to their boundaries," by R. L. MOORE.

THE Council of the Mathematical Association of America has appointed as Committee on Publications Professors R. C. ARCHIBALD (editor in chief) W. A. HURWITZ and H. E. SLAUGHT. This Committee has charge of the official journal of the Association, the *American Mathematical Monthly*.

AT the annual meeting of the Paris academy of sciences, held December 2, 1918, the following prizes in pure and applied mathematics were awarded, in addition to those noted in the December number of the BULLETIN: The Grand prize, for a memoir on the theory of iteration, to G. JULIA, with honorable mention for the late S. LATTÈS, who was also awarded a prize of 2000 francs from the Gegner foundation for his work in mathematical analysis; a prize of 2000 francs from the Jérome Ponti foundation to P. BARBARIN for his work in non-euclidean geometry; a prize of 2000 francs from the Henri Becquerel foundation to P. FATOU for his work on the theory of series and the iteration of rational functions. The Fourneyron, Damoiseau and Guzman prizes were not awarded.

All the prizes are to be awarded in 1919 and following years under the usual conditions. For the Bordin prize (3000 francs), to be awarded in 1921, the following question is proposed: "To improve the theories relating to analysis situs developed in Poincaré's celebrated memoirs. It is required to connect, at least in important special cases, questions of the geometry of situation concerning a given multiplicity with the study of suitably chosen analytic expressions."

DR. F. W. REED has been appointed instructor in mathematics at the University of Illniois.

MR. A. D. CAMPBELL and DR. C. M. SMITH have been appointed instructors in mathematics at Cornell University.

THE resignation of Lieutenant W. E. MILNE, of the Army Ordnance, has been accepted and he has been appointed assistant professor of mathematics at the University of Oregon.

MR. J. S. MIKESH has been appointed instructor in mathematics at Sheffield Scientific School, Yale University.

DR. R. JENTZCH, of the University of Berlin, fell in battle, March 21, 1918.

PROFESSOR W. B. STONE, of Rutgers College, died January 6, 1918. Professor Stone had been a member of the American Mathematical Society since 1913.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BATEMAN (H.). Differential equations. London, Longmans, 1918. 8vo. 11+306 pp. 16s.

BOYER (P.). La vie universitaire à Paris. Paris, Colin, 1918. Fr. 12.00

FORSYTH (A. R.). Solutions of the examples in a treatise on differential equations. London, Macmillan, 1918. 8vo. 249 pp. 10 s.

GALLATLY (W.). The modern geometry of the triangle. 2d edition. London, F. Hodgson, 1918. 8vo. 7+126 pp. 2s. 6d.

HEDRICK (H. B.). Interpolation tables or multiplication tables of decimal fractions giving the products to the nearest unit of all numbers from 1 to 100 by each hundredth from 0.01 to 0.99 and of all numbers from 1 to 1000 by each thousandth from 0.001 to 0.999. Washington, Carnegie Institution, 1918. Folio. 9+139 pp. $5.00

LAUGEL (L.). See SCHWARZ (H. A.).

PLANT (L. C.). See WEBBER (W. P.).

SANSONE (G.). Le divisioni regolari dello spazio iperbolico in piramidi e doppie piramidi. Pisa, tip. succ. fratelli Nistri, 1917. 8vo. 135 pp. con quattro tavole.

SCHWARZ (H. A.). Mélanges relatifs au domaine des surfaces minima. Traduit sur la dernière édition par L. Laugel. Pisa, E. Spoerri, 1918. 8vo. 53 pp. L. 4.00

WEBBER (W. P.) and PLANT (L. C.). Introductory mathematical analysis. New York, Wiley, 1918. 250 pp. $2.00

II. ELEMENTARY MATHEMATICS.

ABBOTT (P.). Mathematical tables and formulæ. London, Longmans, 1918. 4+58 pp. 2s.

——. Numerical trigonometry. London, Longmans, 1918. 10+163+ mathematical tables, 3+33 pp. 5s.

CADORIUS (I.), FISKER (A.), HERLEV (V.) og OLSEN (L. P.). Regnebogen, Elevens Bog IV, V og VI. Lærerens Bog III, IV og V. København, Gyldendalske Boghandel, 1918.

FERRARI (A.). Sopra un capitolo di geometria elementare. Torino, S. Lattes, 1918. 8vo. 32 pp.

FISKER (A.). See CADORIUS (I.).

FRIIS-PETERSEN (F.) og JESSEN (J. L. W.). Realklassens Regnebog. København, Gjellerup, 1918.

GLADDEN (T. L.). Plane geometry. Boston, R. G. Badger, 1918. 165 pp. $1.00

HERLEV (V.). See CADORIUS (I.).

HOPKINS (J. W.) and UNDERWOOD (P. H.). Arithmetic. Book 1, revised. New York, Macmillan, 1918. 12mo. 11+260 pp. $0.40

JESSEN (J. L. W.) og SMITH (O. A.). Matematik og Regning for Realklassen. København, Gjellerup, 1918.

——. See FRIIS-PETERSEN (F.).

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy, the Royal Military College; and papers in elementary engineering for naval cadetships. Nov., 1917 and March, 1918. With answers. London, Macmillan, 1918. 40 pp. 1s. 3d.

MINNICK (J. H.). An investigation of certain abilities fundamental to the study of geometry. (Diss., University of Pennsylvania.) Lancaster, Pa., New Era Press, 1918. Sm. 4to. 8+108 pp. $1.25

NIELSEN (N.). Laerebog i Trigonometri. 2den gennemste Udgave. København og Kristiania, Gyldendalske Boghandel, 1917.

OLSEN (L. P.). See CADORIUS (I.).

SMITH (O. A.). See JESSEN (J. L. W.).

UNDERWOOD (P. H.). See HOPKINS (J. W.).

II. APPLIED MATHEMATICS.

ANDOYER (H.). Formules et tables nouvelles relatives à l'étude du mouvement des comètes et à différents problèmes de la théorie des orbites. Paris, Gauthier-Villars, 1918. 8vo. 52 pp. Fr. 3.00

BONASSE (H.). Astronomie théorique et pratique. (Bibliothèque scientifique de l'ingénieur et du physicien.) Paris, Delagrave, 1918. 8vo. 30+630 pp. Relié. Fr. 35.00

BRITTAIN (W. M.). Bibliography of maritime literature; published by the American steamship association for distribution among the officers and crews of steamers operated by its members. New York, 1918. 8vo. 14 pp.

KEMPNER (A. J.). Simple applications of trigonometry to artillery. Urbana, Ill., University of Illinois, 1918. 8vo. 30 pp. $0.25

LÉVY (M.). La statique graphique et ses applications aux constructions. 3e partie: Arcs métalliques. Ponts suspendus rigides. Conpoles et corps de révolution. 3e édition. Paris, Gauthier-Villars, 1918. 8vo. 10+418 pp. Fr. 24.00

LIPKA (J.). Graphical and mechanical computation. New York, Wiley, 1918. 9+264 pp. $4.00

MAILLOT (L.). Le manuel du mécanicien. Métaux usuels, trempe, recuit, ajustage, tournage, filetage, fraisage, meulage, outils de mesure et de vérification. 3e édition. Paris, Dunod et Pinat, 1918. 8vo. Fr. 3.50

MANN (C. R.). A study of engineering education prepared for the joint committee on engineering education of the national engineering societies. (Bulletin No. 11.) New York, Carnegie Foundation for the Advancement of Teaching, 1918. 4to. 11+139 pp.

MARGUET (F.). Cours d'astronomie de l'Ecole Navale. Paris, Challamel, 1918. 8vo. Fr. 10.00

O'DONNELL (E. E.). Merchant marine manual. Boston, Yachtman's Guide, 1918. 254 pp. $1.00

PALMER (C. I.). Practical mathematics. 2d edition. New York, McGraw-Hill, 1918. 4 volumes. 152+172+188+150 pp. $0.75 each

POOR (C. L.). Simplified navigation for ships and aircraft. A text-book based upon the Saint Hilaire method. New York, Century, 1918. 18+126 pp.

RUST (A.). Practical tables for navigators and aviators, containing new and rapid methods for finding the longitude, azimuth and latitude and for great circle sailing, the identification of stars and for plotting line of position by the Sumner and Marcq Saint Hilaire methods. Philadelphia, J. E. Hand and Sons Company, 1918. 116 pp. $3.50

SALMON (V. G.). Practical surveying and field work. London, C. Griffin and Company, 1918. 7s. 6d.

SAMPSON (R. A.). Studies in clocks and time-keeping. No. 2: Tables of the circular equation. Edinburgh, Grant, 1918. 2s. 10d.

WARNE (E. J.). Elementary course in chartography: twenty lessons, seventy charts. Washington, F. J. Warne, 1917. Paper. $50.00